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Boson operators for generalised harmonic oscillators

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Abstract. A general treatment of generalised quantal harmonic oscillators with time-dependent mass, frequency, damping rate and driving forces has been presented by using boson operators to find the quadratic invariant. Phase factor, eigenstates and state vectors are given. Examples of special cases are also given.

1. Introduction

Consider the Hamiltonian

$$H(t)=\tfrac{1}{2}F(t)p^2/m+\tfrac{1}{2}G(t)m\omega^2x^2+\tfrac{1}{2}J(t)(xp+px)+K(t)x+L(t)p. \qquad (1.1)$$

H describes a time-dependent harmonic oscillator (THO) with time-dependent mass, frequency and, to a certain extent, 'damping' rate denoted by $J(t)$. The last two terms describe the time-dependent driving term.

In this paper we emphasise the importance of finding boson operators $B(t)$ and $B^+(t)$ based on (1.1), which is central to our generalised harmonic oscillator problem. Knowing these operators, we shall find the (Hermitian) invariant $I=\hbar(B^+B+\frac{1}{2})$ for H. Our treatment of the general problem is completely based on the operator formalism which gives directly the solution of the Schrödinger equation corresponding to the Hamiltonian (1.1); it should be mentioned, however, that Lewis (1968) was the first to discuss the Dirac method for finding the eigenstates and eigenvalues for a special case of (1.1).

In the Schrödinger picture operators x and p are independent of time. From a physical consideration, positive mass and frequency are assumed, and therefore $F(t)$ and $G(t)$ are assumed to be positive and real-valued at all time; no such restriction is assumed for $J(t)$, $K(t)$ and $L(t)$ so long as they are real-valued. A compact alternate expression for H is given by

$$H=\tfrac{1}{2}(x,p)A(t)\begin{pmatrix}x\\p\end{pmatrix}+\boldsymbol{b}(t)^{\mathrm{T}}\begin{pmatrix}x\\p\end{pmatrix} \qquad (1.2)$$

where $A(t)$ is assumed to be positive definite for all t. For the Dirac methods as applied to (1.1) or (1.2) see Abdalla and Colegrave (1985) and Abdalla (1986). For brevity the time-dependence notations will be omitted, unless clarity requires otherwise.

2. Boson operators and invariant

We shall define the following operators:

$$B=(2\hbar)^{-1/2}(sx+\mathrm{i}\rho p+\hbar\beta) \qquad B^+=(2\hbar)^{-1/2}(s^*x-\mathrm{i}\rho p+\hbar\beta^*) \quad (2.1)$$

so that the three unknown functions of the time (s, ρ and β) can be determined from an invariant I of H where

$$I = \hbar(B^+B + \tfrac{1}{2}) \tag{2.2a}$$

$$\dot{I} = \frac{\partial I}{\partial t} + \frac{1}{\mathrm{i}\hbar}[I, H] = 0 \tag{2.2b}$$

and the canonical relation

$$[B, B^+] = 1. \tag{2.2c}$$

After considerable algebra, we obtain

$$s = \rho^{-1} + \mathrm{i}\mu(J\rho - \dot{\rho}) \tag{2.3}$$

where ρ satisfies the generalised Pinney equation (Pinney 1950, Lewis 1968, Lewis and Riesenfeld 1969)

$$\ddot{\rho} + \mu^{-1}\dot{\mu}\dot{\rho}[\Omega^2 - \mu^{-1}(\mu \cdot J)]\rho = \mu^{-2}\rho^{-3} \tag{2.4}$$

and

$$\beta = \xi^{-1}\int^t \xi(t')\zeta(t')\,\mathrm{d}t' \qquad \xi = \exp\left(\mathrm{i}\int \mu^{-1}\rho^{-2}\,\mathrm{d}t\right) \tag{2.5a}$$

$$\zeta = \hbar^{-1}[\mathrm{i}(\rho K + \mu\dot{\rho}L) - (\rho^{-1}L + \mu\Omega\rho L)]. \tag{2.5b}$$

The invariant becomes

$$I = \tfrac{1}{2}\{\rho^{-2}x^2 + [\mu(J\rho - \dot{\rho})x + \rho p]^2\} + \hbar[\mathrm{Re}(\beta s^*)x + \mathrm{Im}(\beta)\rho p] + \tfrac{1}{2}\hbar^2\beta^*\beta. \tag{2.6}$$

In the above equations the following parameters are defined:

$$\mu \equiv m/F \tag{2.7}$$

$$\Omega \equiv (FG\omega^2 - J^2)^{1/2} \tag{2.8}$$

where μ is the time-dependent mass and Ω is the shifted frequency due to damping, which is different from $\Omega(t)$ given by Abdalla and Colegrave (1985) and Abdalla (1986). We assume $FG\omega^2 > J^2$ so that only the undercritical damping will be considered.

3. Solution of Schrödinger equation

3.1. Eigenstates of invariant

Lewis and Riesenfeld (1969) showed that the eigenstates of an invariant are solutions of the Schrödinger equation apart from non-global time-dependent phase factors which are significant in the superposition of these states. Since I is invariant, use of the number representation is convenient.

The eigenstates in coordinate representation are

$$u_n(x) = \langle x | n\rangle \qquad B^+|n\rangle = (n+1)^{1/2}|n+1\rangle. \tag{3.1}$$

When standard procedure is used we obtain

$$u_n(x) = A_n \exp\left(-\frac{\rho^{-1} + \mathrm{i}\mu(J\rho - \dot{\rho})}{2\hbar\rho}x^2 - \frac{\beta}{\rho}x - \tfrac{1}{2}\hbar(\mathrm{Re}\,\beta)^2\right)H_n\left(\frac{x}{(\hbar\rho^2)^{1/2}} + \hbar\,\mathrm{Re}\,\beta\right) \tag{3.2a}$$

$$A_n = [2^n n!(\hbar\pi)^{1/2}\rho]^{-1/2} \tag{3.2b}$$

where (3.2) is a generalisation of Lewis and Riesenfeld's (1969) results.

3.2. *Phase factors and time-varying energy*

Lewis and Riesenfeld (1969) showed that it is not $|n\rangle$ but $|n, \alpha_n(t)\rangle = \exp(\mathrm{i}\alpha_n)|n\rangle$ that satisfy the time-dependent Schrödinger equation where α_n is the relative phase to be calculated. In the x representation the eigenvectors and wavefunction are, respectively,

$$\phi_n(x, t) = \exp(\mathrm{i}\alpha_n)u_n(x) \qquad \psi(x, t) = \sum_n C_n\phi_n(x, t) \tag{3.3}$$

when α_n satisfies

$$\hbar\dot{\alpha}_n = \langle n|\mathrm{i}\hbar\partial/\partial t - H|n\rangle. \tag{3.4}$$

Since $|n\rangle$ are the eigenstates of I and of the number operator B^+B, the diagonal matrix elements $\langle n|H|n\rangle$ can be evaluated as

$$\begin{aligned} &\varepsilon_n(t) = \langle n|H|n\rangle = (\hbar/2\mu)(n+2^{-1})[\rho^{-2}+\mu^2\dot{\rho}^2+(\mu\Omega\rho)^2]+(\hbar^2/2)F_1(t)-\hbar F_2(t)\\ &F_1(t) = \mu(\Omega\rho\beta_{\mathrm{r}})^2+\mu^{-1}(\mu\dot{\rho}\beta_{\mathrm{r}}+\rho^{-1}\beta_{\mathrm{i}})^2\\ &F_2(t) = \rho\beta_{\mathrm{r}}(K-\mu JL)+(\mu\dot{\rho}\beta_{\mathrm{r}}+\rho^{-1}\beta_{\mathrm{i}})L \end{aligned} \tag{3.5}$$

where $\beta_{\mathrm{r}} = \mathrm{Re}\,\beta$ and $\beta_{\mathrm{i}} = \mathrm{Im}\,\beta$. Notice that when $K = L = 0$ or $\beta = 0$ (3.5) reduces to that of Lewis and Riesenfeld (1969).

To evaluate $\langle n|\partial/\partial t|n\rangle$ we follow Lewis and Riesenfeld closely. Their equations (53) and (56) can be used here provided their a and a^+ are replaced by B and B^+. Taking the time derivative of B^+ and using their arguments about the phase choice, we obtain

$$\langle n|\partial/\partial t|n\rangle = (\mathrm{i}\mu/2)(n+2^{-1})D \tag{3.6a}$$

where

$$D = \rho\ddot{\rho}-\dot{\rho}^2+\mu^{-1}\dot{\mu}\rho\dot{\rho}-\mu^{-1}(\mu\cdot J)\rho^2. \tag{3.6b}$$

Substituting (3.6) and (3.5) into (3.4), we obtain

$$\alpha_n = -(n+2^{-1})\int^t \mu^{-1}(t')\rho^{-2}(t')\,\mathrm{d}t' - (\hbar/2)\int^t F_1(t')\,\mathrm{d}t' + \int^t F_2(t')\,\mathrm{d}t' \tag{3.7}$$

where, in obtaining the first integral, (2.4) has been used. For the special case where $\mu = m$ and $K = L = F_1 = F_2 = 0$ then (3.7) reduces to that obtained by Lewis and Riesenfeld (1969).

It is interesting to note that $-\alpha_n$ may be interpreted as being proportional to energy $\bar{\varepsilon}_n(t)$ where

$$\bar{\varepsilon}_n = \hbar\bar{\omega}_n \qquad \bar{\omega}_n(t) = (n+2^{-1})\mu^{-1}\rho^{-2}+\hbar F_1/2-F_2.$$

This energy, though time varying, is different from $\varepsilon_n(t)$ in (3.5) for general time-dependent THO. For time-invariant HO, however, they coincide. To see this and other aspects, we set $\bar{\varepsilon}_n(t) = -\hbar\dot{\alpha}_n(t)$ in (3.4), define $\Delta\varepsilon_n = \varepsilon_n - \bar{\varepsilon}_n$ and obtain

$$\Delta\varepsilon_n = \langle n|\mathrm{i}\hbar\partial/\partial t|n\rangle = (\hbar/2\mu)(n+2^{-1})[\mu^2\dot{\rho}^2+(\mu\Omega\rho)^2-\rho^{-2}]. \tag{3.8}$$

This implies that $\Delta\varepsilon_n$ is a measure of the time dependence of eigenstates $|n\rangle$. If $|n\rangle$ does not depend on time then $\Delta\varepsilon_n = 0$ implying a time-invariant HO system in which $\dot{\rho} = 0$ and, from (3.8), $\rho = (\mu\Omega)^{-1/2}$ = constant which also satisfies (2.4) as it should. In this case $\bar{\varepsilon}_n = \varepsilon_n = -\hbar\dot{\alpha}_n$ = constant or $\alpha_n = -\varepsilon_n t/\hbar$ and the state vector $\phi_n(x, t) = \exp(-\mathrm{i}\varepsilon_n t/\hbar)u_n(x)$, a well known result.

We prove next that $|n, \alpha\rangle$ satisfies the Schrödinger equation. Let $|n\rangle = \exp(-\mathrm{i}\alpha_n)|n, \alpha_n\rangle$ so that $\langle n| = \langle n, \alpha_n|\exp(\mathrm{i}\alpha_n)$. Then (3.4) reduces to

$$(\mathrm{i}\hbar\partial/\partial t)|n, \alpha_n\rangle = H|n, \alpha_n\rangle \tag{3.9a}$$

$$|n, \alpha_n\rangle = \exp(\mathrm{i}\alpha_n)|n\rangle. \tag{3.9b}$$

Clearly $|n\rangle$ does not satisfy the Schrödinger equation. However, $|n\rangle$ satisfies an equation analogous to (3.9) in which the energy operator is changed

$$\mathrm{i}\hbar\partial/\partial t \to \mathrm{i}\hbar\partial/\partial t - \hbar\dot{\alpha}_n = \mathrm{i}\hbar \exp(-\mathrm{i}\alpha_n)(\partial/\partial t)\exp(\mathrm{i}\alpha_n) \tag{3.10}$$

or, using $\langle n|n\rangle = 1$ in (3.4), we have

$$\mathrm{i}\hbar \exp(-\mathrm{i}\alpha_n)(\partial/\partial t)\exp(\mathrm{i}\alpha_n)|n\rangle = H|n\rangle. \tag{3.11}$$

When H and thus $|n\rangle$ and $\dot{\alpha}_n$ do not depend on t, (3.11) reduces to the well known stationary eigenvalue equation $-\hbar\dot{\alpha}_n|n\rangle = \varepsilon_n|n\rangle = H|n\rangle$.

4. Examples

If we make the following changes:

$$F = m/M(t) \qquad G = (M(t)/m)\omega^2(t)/\omega^2 \tag{4.1}$$

then

$$\mu = M(t) \qquad \Omega^2 = \omega^2(t) - J^2 \qquad \kappa = \dot{\mu}/\mu = \dot{M}(t)/M(t). \tag{4.2}$$

The Hamiltonian (1.1) without the driving terms is now expressed in the conventional form

$$H_0 = \tfrac{1}{2}p^2/M(t) + \tfrac{1}{2}M(t)\omega^2(t)x^2 + \tfrac{1}{2}J(t)(xp + px) \tag{4.3}$$

and the Pinney equation (2.4) can be rewritten as

$$\ddot{\rho} + \kappa\dot{\rho} + [\omega^2(t) - (J^2 + \dot{J} + \kappa J)]\rho = M^{-2}(t)\rho^{-3}. \tag{4.4}$$

Letting $\rho = M^{-1/2}(t)y$ in (4.4), we obtain

$$\ddot{y} + \bar{\Omega}^2(t)y = y^{-3} \tag{4.5}$$

$$\bar{\Omega}^2(t) = \omega^2(t) - (\tfrac{1}{2}\dot{\kappa} + \tfrac{1}{4}\kappa^2 + \dot{J} + \kappa J + J^2). \tag{4.6}$$

Thus the formulae, for obtaining ρ, of Pinney (1950), Eliezer and Gray (1976), Leach (1983) and Lewis (1968) apply.

In the following examples we set $J = K = L = 0$, hence $\beta = 0$ and $\Omega = \omega(t)$. The phase angle for each is equal to the first integral in (3.7).

4.1. *Time-varying frequency only*

Setting $M(t) = m$ and $\omega(t) = \bar{\Omega}(t)/m$ in (4.3), then (4.4) and (3.2) give

$$H = [p^2 + \bar{\Omega}^2(t)x^2]/2m \qquad m^2\ddot{\rho} + \bar{\Omega}^2(t)\rho = \rho^{-3}$$

$$u_n(x) = A_n \exp\left(-\frac{\rho^{-1} - \mathrm{i}m\dot{\rho}}{2\hbar\rho}x^2\right)H_n\left(\frac{x}{\rho\hbar^{1/2}}\right) \tag{4.7}$$

in agreement with Lewis (1968). We recover his invariant from (2.6)

$$I=\tfrac{1}{2}[\rho^{-2}x^2+(\rho p-m\dot{\rho}x)^2]$$

(see also Leach 1977, Colegrave and Abdalla 1983).

4.2. *Time-varying mass only*

Setting $\omega(t)=\omega$ we have

$$H=p^2/2M(t)+\tfrac{1}{2}M(t)\omega^2x^2 \qquad \ddot{\rho}+\kappa\dot{\rho}+\omega^2\rho=M^{-2}(t)\rho^{-3} \tag{4.8}$$

$$u_n(x)=A_n\exp\left(-\frac{1-\mathrm{i}M(t)\rho\dot{\rho}}{2\hbar\rho^2}x^2\right)H_n\left(\frac{x}{\rho\hbar^{1/2}}\right) \tag{4.9}$$

which is in agreement with Leach (1983). The invariant is

$$I=\tfrac{1}{2}[\rho^{-2}x^2+(\rho p-M(t)\dot{\rho}x)^2]. \tag{4.10}$$

4.3. *Time-varying mass and frequency*

Setting $J=0$ in (4.3) and (4.4) we have

$$H=p^2/2M(t)+\tfrac{1}{2}M(t)\omega^2(t)x^2 \qquad \ddot{\rho}+\kappa\dot{\rho}+\omega^2(t)\rho=M^{-2}(t)\rho^{-3}. \tag{4.11}$$

$u_n(x)$ and I have the same forms as those in § 4.2.

4.4. *Strongly pulsating oscillator*

Let $\mu=M(t)=m\cos^2(\nu t)$ in (4.2), then we have $\kappa=-2\nu\tan(\nu t)$. The Pinney equation (4.8) is given by

$$\ddot{\rho}-2\nu\tan(\nu t)\dot{\rho}+\omega^2\rho=m^{-2}\cos^{-4}\nu t\rho^{-3}. \tag{4.12}$$

It can be easily shown (Eliezer and Gray 1976) that

$$\rho=\sec(\nu t)/(m\lambda)^{1/2} \tag{4.13a}$$

$$\lambda=(\omega^2+\nu^2)^{1/2}. \tag{4.13b}$$

The invariant I in (4.10) becomes

$$I=\tfrac{1}{2}C_1p^2+\tfrac{1}{2}C_2x^2+\tfrac{1}{2}C_3(xp+px) \tag{4.14}$$

where

$$C_1=\rho^2=\sec^2(\nu t)/m\lambda \tag{4.15a}$$

$$C_2=\rho^{-2}+M^2(t)\dot{\rho}^2=(m/\lambda)(\omega^2\cos^2(\nu t)+\nu^2) \tag{4.15b}$$

$$C_3=-\rho\dot{\rho}M(t)=-\nu\tan(\nu t)/\lambda. \tag{4.15c}$$

When (4.15) is substituted into (4.14) I is seen to be the same as the invariant (55) of Abdalla and Colegrave (1985) except for the unimportant constant factor λ and with $Q=x\cos(\nu t)$ and $P=p\sec(\nu t)$.

5. Conclusion

The quadratic invariant of the generalised time-dependent harmonic oscillator has been obtained by using boson operators without the generator formalism. A generalised Pinney equation and phase factor formulae have also been obtained. Eigenstates and state vectors are given for time-varying mass, frequency, damping rate and driving forces. The solutions of the Schrödinger equation so obtained provide useful alternatives to those obtained using either the normal-ordering operators method or the time-dependent perturbation method. We have presented the diagonal matrix element of the Hamiltonian. Its off-diagonal counterpart can be obtained without difficulty together with matrix elements of x, p, x^2 and p^2. (For time-dependent mass, see Leach (1983).) The propagator without the Maslov correction factor for THO has been presented by Landovitz *et al* (1983). Cheng (1985) gives the propagator of THO with a time-dependent mass, including the Maslov correction factor. We have not discussed in this paper the linear invariants (Colegrave and Abdalla 1983) which are non-Hermitian but are useful for constructing coherent states (Malkin *et al* 1973).

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